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MESH EFFECTS ON THE COMPUTATION OF SMALL AMPLITUDE WATER WAVES

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SUMMARY

Computational difficulties arise in the non-linear free-surface problem for water waves both at large amplitudes when the crest becomes nearly singular and at small amplitudes when the wave is very close to the alternative uniform flow solution. Since the limiting wavelengths for small amplitude waves are known from the Stokes linearized theory, these are used in checking results for finite-amplitude programs. When Southwell and Vaisey[1] first tried this, their methods gave an unexplained overestimate, by 6 per cent, of the limiting wavelength. This paper shows how coarse mesh effects can create such an overestimate, gives very accurate solutions at small amplitudes and considers accuracy in relation to the mesh for short and long waves.

INTRODUCTION

This paper is concerned with surface 'water' waves under gravity represented by two-dimensional ideal flow in a vertical plane. For the purpose of computation, travelling waves of permanent form are best considered in the frame of reference in which they are steady. They may then be regarded as stationary features on a moving stream with a fixed total volume flow rate Q (per unit width of the stream) and a fixed total head H. We shall consider only waves over a flat horizontal bed so that H will be the height of the stagnation level above the bed. In the following all quantities will be non-dimensionalized with respect to distance H and time $(H/g)^{1/2}$ where g is the gravitational acceleration. Consequently H and g will not appear explicitly and 1 (unity) will be in their place.

Various computational difficulties arise in the non-linear free-surface problem for water waves. At large amplitudes the crest may become nearly singular and requires careful resolution. At small amplitudes the wave solution competes with the alternative uniform flow in the computational process. Long waves of small amplitude carry an extra difficulty in that they are close to a uniform critical flow, where a singularity arises in the governing system and iteration methods.[2]

Numerical techniques capable of computing both linear and non-linear waves have been presented by several authors. Young[3] gives a comprehensive review paper on subject. Some prominent developments have been due to Longuet-Higgins[4–6] who has, among other techniques, used boundary integral methods. Schwartz[7] introduced the use of Pade approximants

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in his higher-order perturbation method. Cokelet[8] extended this technique to cover a wider range of wave amplitudes and depths in his accurate numerical computations. Accuracy is reported to deteriorate for the higher waves in shallow water and convergence may fail. Williams[9] has recently presented a technique to compute the highest waves together with a set of accurately computed results, but the algorithm cannot explicitly compute solitary waves. More recently Hunter and Vanden-Broeck[10] have presented accurate computations for high solitary waves confirming and extending previous results. The present authors have also obtained solutions for a wide range of small and large amplitude waves including conoidal and solitary waves but not cusped waves, using finite element and streamline shifting methods. However, the present paper is particularly concerned with the numerical difficulties encountered for small amplitude waves.

Small amplitude waves can be treated by a first order perturbation to give the Stokesian waves theoretical results.[11,12] These are very useful in providing a check for computational methods. The results give a uniquely defined wavelength λ_0 for limiting small amplitude waves, depending on the flow depth h:

$$\lambda_0 = 4\pi(1-h)/\tanh(2\pi h/\lambda_0). \tag{1}$$

The flow depth h is the depth of the subcritical uniform stream with flow rate Q. In the limit as wave amplitude tends to zero this is the same as the average flow depth[1,12] and the still water depth,[8] which are used for reference with finite amplitude waves where they differ. Then h is the larger solution in $(0, 1)$ to

$$Q^2 = 2h^2(1-h) \tag{2}$$

and the wave speed relative to 'still water' in this limiting small-amplitude case and the non-dimensionalized units is

$$\begin{aligned} c^2 &= Q^2/h^2 = 2(1-h) \\ &= (\lambda_0/2\pi)\tanh(2\pi h/\lambda_0). \end{aligned} \tag{3}$$

On redimensionalizing, this last equation gives the celebrated formula for c^2 in terms of g, h and λ_0, not involving H. The perturbation methods have been extended, particularly by Cokelet,[8] to large amplitude waves. Much earlier Southwell and Vaisey[1] gave numerical results for waves and found an unexplained 6 per cent error in extrapolation of their results to compare with the limiting λ_0 in (1). There have been several computational results since, for example using finite element and boundary element methods;[13,14] but the error of Southwell and Vaisey[1] has remained unexplained.

Toro[15] made small-amplitude computations by both finite elements and a streamline shifting method, and examined the effects of mesh size on the extrapolation to λ_0. Here we present and extend these results to show how mesh coarseness leads to errors of the kind found by Southwell and Vaisey, overestimating λ_0 for short waves, and of the opposite type for long waves. We also give results with very accurate agreement with λ_0 using a simple mesh with only a few layers, obtained from our variational streamline-shifting method.[16]

COMPUTATIONAL DETAILS

Recent finite element methods for ideal open channel flows[13,17] have used a governing variational principle[18] of the form (in non-dimensionalized units)

$$J = \int\int_R \left(\tfrac{1}{2}(\nabla\psi)^2 - y\right)\mathrm{d}x\,\mathrm{d}y = \text{stationary}, \tag{4}$$

in terms of the stream function ψ over the flow region R where x and y are horizontal and vertical co-ordinates. This gives both the internal flow and the free surface position by its natural stationary conditions. The authors have also used this approach successfully both with finite elements and with a new streamline shifting algorithm.[16]

The corresponding free-boundary problem for ψ comprises Laplace's equation for ψ in R, prescribed values for ψ on the free surface and on the bed, appropriate boundary conditions on the inlet and outlet boundaries, and the free-surface pressure condition. In computing waves which are symmetric about their crests and troughs, the inlet and outlet conditions may be taken as requiring the flow to be horizontal. That is the normal derivative $\partial\psi/\partial n$ is taken to be zero on vertical inlet and outlet boundaries, which are taken to be at a crest and trough separated by half a wavelength.

The variational formulation makes the problem amenable to computation by introducing a variable mesh able to fit to a variable position for the free surface. Then, apart from the prescribed values for ψ on the bed and free surface, all the required equations are satisfied as natural conditions for the variational principle.

The approach we have taken to find a wave solution is to set the length L of the computing region (inlet to outlet) suitably near half the known limiting wavelength λ_0 for the prescribed flow rate Q, and then to start the iteration for the non-linear equations at an initial half-wave approximation for the free-surface. In the course of the iterations the computed free-surface position adapts itself to satisfy the non-linear problem. If L has been appropriately set, the iterations converge to a half wave of a particular amplitude A related to the wavelength $\lambda = 2L$ and Q. In some cases, if the initial half-wave approximation is poorly chosen, or if L is set out of range of wavy solutions for the given Q, then the computed free-surface position usually 'slips' into the uniform flow solution. In a few cases the crest and trough have changed places during the iterative process.

By contrast Southwell and Vaisey[1] used finite differences without the variational approach, and they set the wave amplitude and allowed the wavelength to adjust to it by stretching the mesh horizontally.

Both our calculations and those of Southwell and Vaisey lead to sets of values for (A, λ) for a given Q and hence for a given h by (2). These are then extrapolated as $A \to 0$ to obtain a computed value for λ_0. Different (A, λ) curves are obtained from different meshes. On a given mesh the (A, λ) curve approaches $A = 0$ with $\mathrm{d}\lambda/\mathrm{d}A$ also tending to zero, compatibly with a smooth transition to negative values of 'amplitude' as the crest changes into a trough. A parabolic relationship of the form $\lambda = \lambda_0 + kA^2$ fits the computed points well in the vicinity of $A = 0$ and the corresponding extrapolation

$$\lambda_0 = (\lambda_1 A_2^2 - \lambda_2 A_1^2)/(A_2^2 - A_1^2) \tag{5}$$

is used to estimate λ_0 from two neighbouring computed points (A_1, λ_1), (A_2, λ_2).

The first order perturbation field from the uniform flow has stream function as in the Stokes theory,

$$\psi = -C \sin(2\pi x/\lambda) \sinh(2\pi y/\lambda). \tag{6}$$

Consequently perturbation values, gradients and velocities, and truncation errors in the discretization of both $\nabla^2\psi$ and of streamline positions, are all greatest at the surface, diminishing towards the bed. The variation from surface to bed is very pronounced for short waves with $\lambda \ll h$ and it falls off exponentially from the surface. Mesh resolution is then important near the surface but not in the bulk of the region. The variation becomes less pronounced for longer waves and for $\lambda \gg h$ the flow becomes nearly independent of y so that little vertical resolution is then needed. Indeed a single discrete stream layer gives very accurate results in the region of 'shallow water' waves.

In our finite element and moving streamline general methods we have obtained accurate results over a range of wavelengths using up to about eight flow layers of equal depths (for finite elements) or equal flow rate (for moving streamlines), without introducing surface refinements.

Discretization errors in streamline interpolation on a variable grid will depend on velocity gradients. Discretization errors in $\nabla^2\psi$ will depend on the fourth derivatives of ψ. In either case for fields of type (6) the contributions to these errors arising from discretization in x and y will have approximately equal bounds when the mesh spacings in x and y directions are equal to each other. Thus an optimal mesh will have spacings $h_x \approx h_y$, whether for long and short waves. For example on a rectangular mesh the dominant error in the common 5-point approximation to $\nabla^2\psi$ is, using (6),

$$\begin{aligned} e &= -(h_x^2\partial^4\psi/\partial x^4 + h_y^2\partial^4\psi/\partial y^4)/12 \\ &= -(2\pi/\lambda)^4(h_x^2 + h_y^2)\psi/12. \end{aligned}$$

On a non-rectangular mesh the condition $h_x \approx h_y$ needs to be interpreted in a local sense. The meshes we use, which are commonly used in such problems, are in any case quasi-rectangular, following the streamlines (roughly for finite elements, precisely for streamline shifting) with vertical transversals.

COMPUTATIONAL EXAMPLES

Example 1

The first example is the short wave case treated by Southwell and Vaisey[1] with their 6 per cent overestimate for λ_0. The flow depth, relative to the head H, is 11/12 and correspondingly $Q^2 = 0{\cdot}1400463$ and $\lambda_0 = 1{\cdot}04723$ by the Stokesian theory.[1]

Southwell and Vaisey gave their extrapolated solution as $\lambda_0 = 1{\cdot}11$, declaring that they had not succeeded in accounting for the discrepancy.

Unusually they did not identify their mesh details for the calculation except for the number of stations they used. From their Figure 23 it would appear as if they took $M = 14$ for the longer waves and $M = 10$ for the shortest (highest) wave. It is possible that their result is affected by distorted meshes and inadequate surface resolution. Their highest wave has $\lambda = 0{\cdot}88$ whereas our recent computations give $\lambda = 0{\cdot}8811093$ using 72 layers and 40 stations, which is about 1 per cent higher than the result we would obtain by interpolating Cokelet's results. Hence their results are more accurate for the highest wave.

The worsening of the solution by increasing the number M of stations (Figure 2) may be due to distortions of the rectangular 'elements'. For the smaller amplitude waves $dy \approx 1{\cdot}1833$, whereas dx ranges from 0·218 down to 0·1211, The exponential variation of the horizontal velocity profile for this example in the short wave region requires better resolution in the y-direction (i.e. $dy \leqslant dx$) than in the x-direction. Thus it is possible that coarser meshes give better results provided that the right mesh ratios are observed.

Figure 1 shows Southwell and Vaisey's results together with several sets of our results from meshes with various numbers of discrete layers but a fixed number, 6, of stations in the x-direction. The 6 stations correspond to 5 mesh spaces over the half wave, so $h_x = \lambda/10 \approx 0{\cdot}1$ in the x-direction. This is adequate for accuracy to within half a per cent on λ_0. Comparable accuracy is then obtained from a similar y spacing which corresponds to about 9 layers. Taking 8 layers provides a very accurate result, extrapolating to 1·0489. Taking fewer layers increases the error of overestimation substantially.

In Figure 2 some A, λ curves are given for a fixed number of layers, 5, which is adequate only for 3

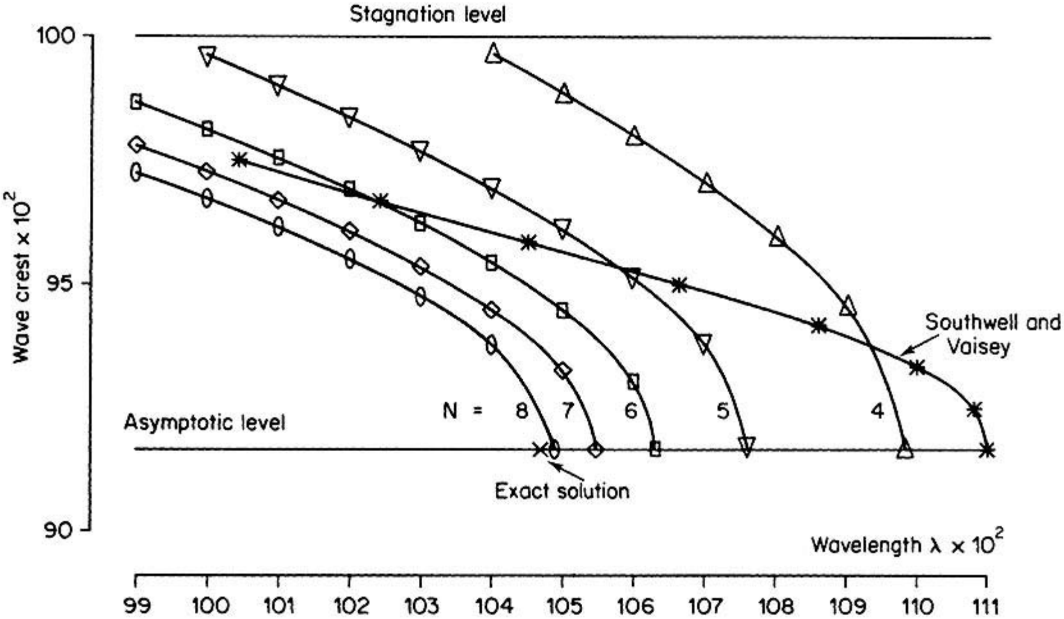

Figure 1. Variation of the computed (A, λ) curves with the number N of layers and fixed number M of stations ($M = 6$)

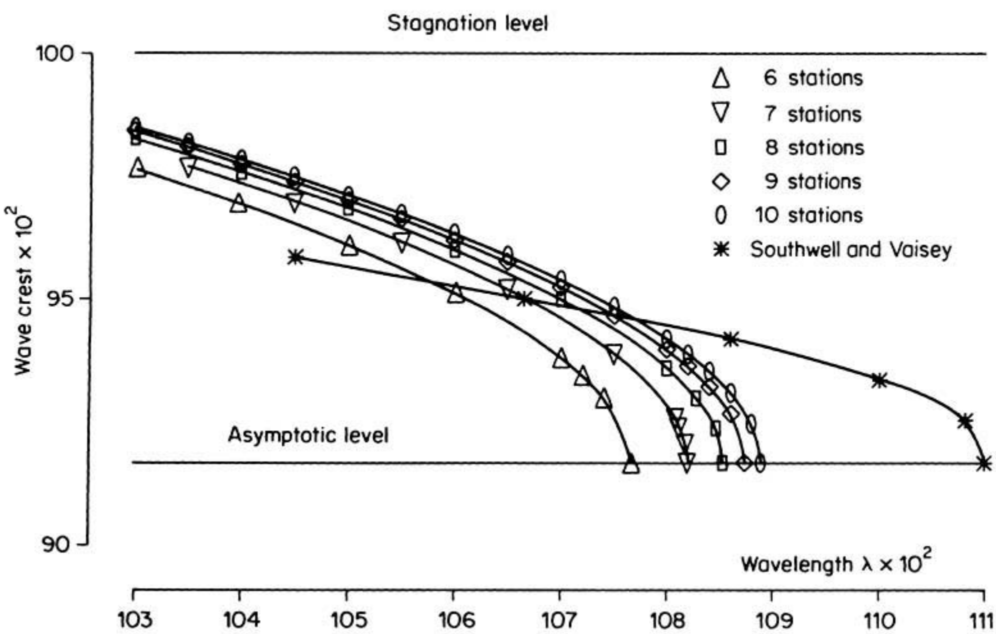

Figure 2. Variation of the computed (A, λ) curves with the number M of stations and fixed number N of layers ($N = 5$)

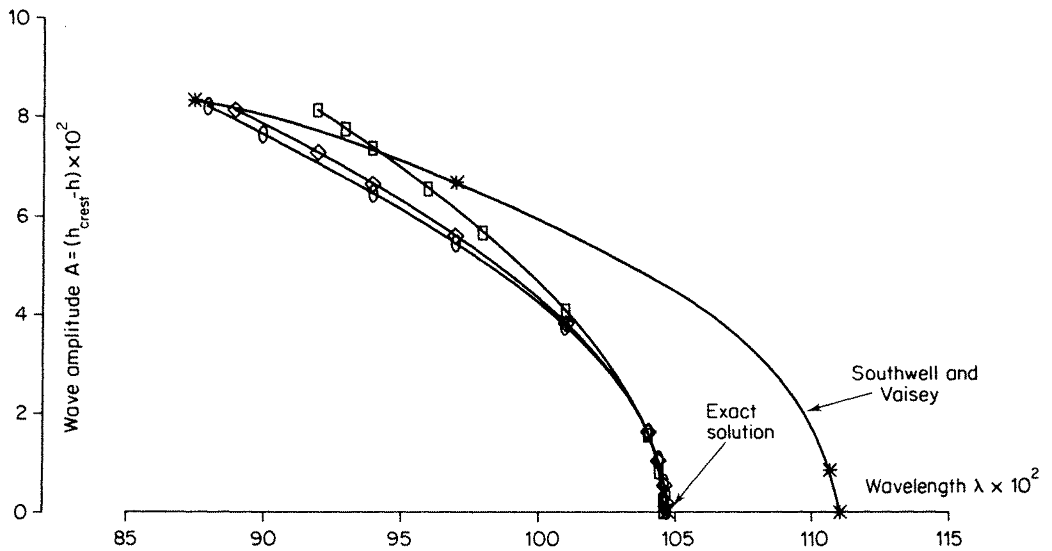

Figure 3. Effect of mesh refinement on the computed (A, λ) curve retaining the same space ratio

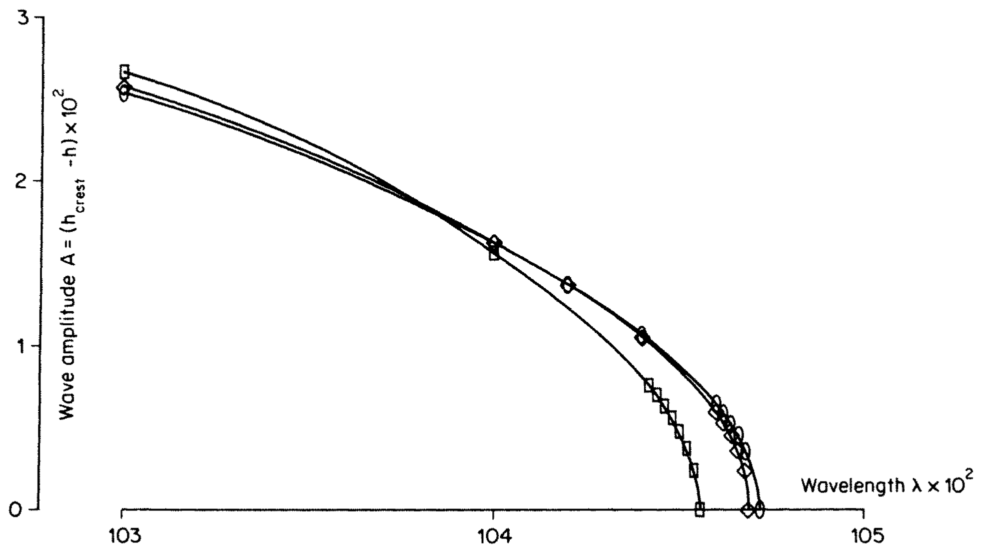

Figure 4. Effect of mesh refinement on the computed (A, λ) curve near λ_0 retaining the same space ratio

or 4 per cent accuracy. Increasing the numbers of stations does not compensate for this and in fact tends to increase the estimate for λ_0.

The effect of refining the mesh while retaining the same space ratio is shown in Figures 3 and 4. Starting with an accurate and reasonably balanced mesh of 9 layers and 6 stations, that is $h_x \approx h_y \approx 0{\cdot}1$, this is halved to 18 layers and 11 stations and halved again to 36 layers and 21 stations. The very accurate results are listed in Table I.

Table I Results for λ_0 for halved meshes for Example 1 (exact solution $\lambda_0 = 1{\cdot}04723$)

No. of stations	No. of layers	Extra-polation for λ_0	percentage
6	9	1·04553	0·162
11	18	1·04695	0·027
21	36	1·04717	0·006
extrapolation to ∞		1·04721	0·002

The extrapolations for λ_0 in Table I are made from actual computations for λ very close to λ_0, within 0·0004 with amplitudes below 0·005. The last row, extrapolation to infinity, extrapolates from the halving process assuming an error dependence on mesh space s of the form

$$\lambda(s) = \lambda_0 + ks^{\alpha}. \tag{7}$$

Then differencing and dividing the results of Table I gives

$$\begin{aligned} 2^{\alpha} &= (\lambda(s/2) - \lambda(s/4))/(\lambda(s) - \lambda(s/2)) \\ &= 0{\cdot}00142/0{\cdot}00022, \end{aligned}$$

giving an index $\alpha = \ln(142/22)/\ln 2 \approx 2{\cdot}69$, rather better than the $\alpha = 2$ which might be expected from the linear elements or stream layers used. Consequently an extrapolated λ_0 as mesh size tends to zero is obtained from (7) as

$$\begin{aligned} \lambda_0 &= 1{\cdot}04717 + (22/142)(142/120)0{\cdot}00022 \\ &= 1{\cdot}04721 \end{aligned}$$

This is still very slightly below the exact value, so perhaps the error is asymptotically like $\alpha = 2$, not $\alpha = 2{\cdot}69$ as from the three finite meshes. Extrapolating at $\alpha = 2$ from the two finest grids would give $\lambda_0 = 1{\cdot}04724$.

Example 2

This example gives a wavelength in the conoidal range, with $h = 0{\cdot}6839397$, $Q^2 = 0{\cdot}2956893$ and $\lambda_0 = 8{\cdot}5948397$. Figure 5 shows a set of (A, λ) points calculated from a mesh of 4 layers and 10 stations, with $h_x \approx 2$ and $h_y \approx 0{\cdot}2$ which is relatively coarse in the x-direction. Nevertheless, an accuracy of about 1 per cent is obtained for λ_0. Notice now that λ increases with A, whereas for short waves λ decreases with A (for waves of moderate height).

Example 3

We provide here a long shallow water example, since for small amplitudes these encounter the difficulty of the singularity of critical flow. This case has $h = 0{\cdot}6674855$ (the critical value is 2/3), $Q^2 = 0{\cdot}296295$ and $\lambda_0 = 39{\cdot}8013064$. Three meshes were used, 90 stations and 3 layers, 180 stations and 6 layers, and 360 stations and 12 layers, with virtually identical results for (A, λ) to five significant figures, as might be expected since we now have 90 or more divisions per half wave and $h_y \approx h_x$. Figure 6 shows the (A, λ) curve obtained which gives λ_0 with about 0·1 per cent error. Convergence difficulties were encountered at values of λ very close to λ_0, but computations

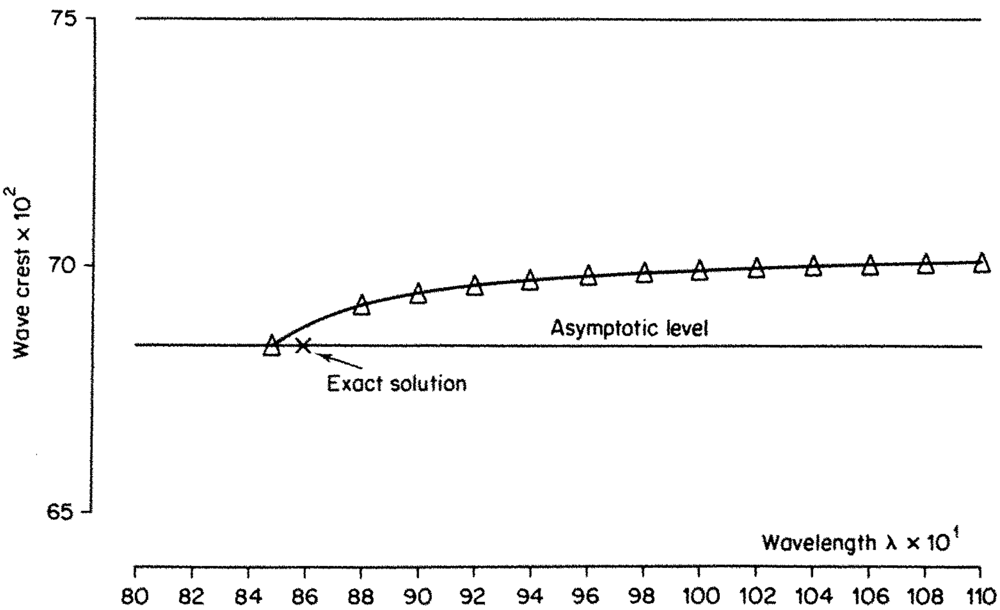

Figure 5. Computed (A, λ) curve for mesh of 5 layers and 10 stations for example 2 in conoidal region

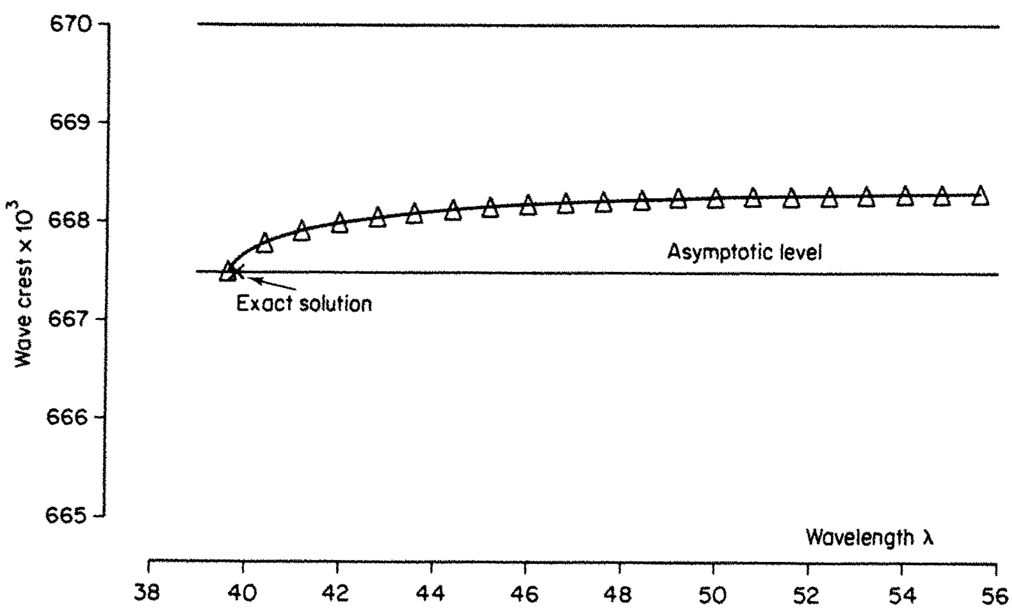

Figure 6. Computed (A, λ) curve for 3 layers and 180 stations for example 3 in the shallow water region

worked down to $\lambda = 40{\cdot}4$ with amplitude as small as $A = 0{\cdot}0003$. Results almost as accurate were obtained with a single-layer, 30 station mesh for this shallow water wave.

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